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# On a family of integrable systems on $S^2$ with a cubic integral of motion

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**Abstract**
We discuss a family of integrable systems on the sphere $S^2$ with an additional third-order integral in momenta. This family contains the Coryachev–Chaplygin top, the Goryachev system, the system recently discovered by Dullin and Matveev, and two new integrable systems. On the non-physical sphere with zero radius all these systems are isomorphic to each other.

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## 1. Introduction

Let us consider a particle moving on the sphere $S^2 = \{x \in \mathbb{R}^3, |x| = a\}$. As coordinates on the phase space $T^*S^2$ we choose entries of the vector $x = (x_1, x_2, x_3)$ and entries of the angular momentum vector $J = p \times x = (J_1, J_2, J_3)$, where $J_i = \sum \varepsilon_{ijk} p_j x_k$. The corresponding Poisson brackets read

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \qquad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \qquad \{x_i, x_j\} = 0, \tag{1.1}$$

where $\varepsilon_{ijk}$ is the totally skew-symmetric tensor. The Casimir functions of the brackets (1.1)

$$A = \sum_{i=1}^{3} x_i^2 = a^2, \qquad B = \sum_{i=1}^{3} x_i J_i = 0, \tag{1.2}$$

are in the involution with any function on $T^*S^2$. So, for the Liouville integrability of the corresponding equations of motion it is enough to find only one additional integral of motion, which is functionally independent of the Hamiltonian $H$ and the Casimir functions.

If the corresponding Hamilton function $H$ has a natural form, then according to Maupertuis's principle, integrable system on $T^*S^2$ immediately gives a family of integrable geodesic on $S^2$. If the additional integral of this integrable system is polynomial in momenta, integral of the geodesic are also polynomial of the same degree.

  

In this note we discuss a family of integrable systems on $T^*S^2$ with a cubic additional integral of motion. Among such systems we distinguish the Goryachev–Chaplygin top [1, 2] with the following integrals of motion

$$H = J_1^2 + J_2^2 + 4J_3^2 + cx_1, \qquad K = 2\big(J_1^2 + J_2^2\big)J_3 - cx_3J_1, \qquad c \in \mathbb{R}. \tag{1.3}$$

In [2] Chaplygin found the separated variables

$$\mathrm{q}_j = J_3 \pm \sqrt{J_1^2 + J_2^2 + J_3^2}, \qquad j = 1, 2, \tag{1.4}$$

where dynamical equations are equal to

$$(-1)^j(\mathrm{q}_1 - \mathrm{q}_2)\dot{\mathrm{q}}_j = 2\sqrt{P(\mathrm{q}_j)^2 - a^2c^2\mathrm{q}_j^2}, \qquad P(\lambda) = \lambda^3 - \lambda H + K. \tag{1.5}$$

These equations are reduced to the Abel–Jacobi equations and, therefore, they are solved in quadratures [2].

Using variables $\mathrm{q}_j$ (1.4), integrals of motion (1.3) may be rewritten in the following form:

$$H = \mathrm{q}_1^2 + \mathrm{q}_1\mathrm{q}_2 + \mathrm{q}_2^2 + cx_1, \qquad K = \mathrm{q}_1\mathrm{q}_2(\mathrm{q}_1 + \mathrm{q}_2) - cx_3J_1. \tag{1.6}$$

In [3] Goryachev *de facto* substituted special generalizations of the variables $\mathrm{q}_j$ (1.4) into expressions similar to (1.6) in order to construct new integrable system with a cubic integral of motion. In the next section we generalize this result.

## 2. A family of integrable systems on the sphere

Substituting canonical variables

$$q_j = \alpha J_3 \pm \sqrt{J_1^2 + J_2^2 + f(x_3)J_3^2 + g(x_3)}, \qquad \{q_1, q_2\} = 0, \tag{2.1}$$

into the following ansatz for integrals of motion:

$$\begin{aligned} H &= q_1^2 + q_1q_2 + q_2^2 + m(x_3)x_1, \\ K &= q_1q_2(q_1 + q_2) - n(x_3)J_1 - \ell(x_3)x_1J_3, \end{aligned} \tag{2.2}$$

one gets

$$H = J_1^2 + J_2^2 + (3\alpha^2 + f(x_3))J_3^2 + m(x_3)x_1 + g(x_3) \tag{2.3}$$

and

$$K = -2\alpha J_3\big(-\alpha^2J_3^2 + J_1^2 + J_2^2 + f(x_3)J_3^2 + g(x_3)\big) - n(x_3)J_1 - \ell(x_3)x_1J_3. \tag{2.4}$$

Here $\alpha$ is an arbitrary numerical parameter, $f, g, m, n$ and $\ell$ are some functions of $x_3$ and of the single non-trivial Casimir $a = \sqrt{x_1^2 + x_2^2 + x_3^2}$ (1.2).

**Theorem 1.** *On the phase space $T^*S^2$ functions $H$ (2.3) and $K$ (2.4) are in the involution with respect to the brackets (1.1) if and only if function $n(x_3)$ is a solution of the following differential equation depending on $\alpha^2$*

$$\begin{aligned} 24\alpha^2 - 9 = 15\frac{x_3n' - n''\big(a^2 - x_3^2\big)}{n} + \frac{3x_3n'' - n'''\big(a^2 - x_3^2\big)}{n'}\left(9 - \frac{nn''}{n'^2}\right) \\ + n\left(\frac{5x_3n''' - n''''\big(a^2 - x_3^2\big) + 3n''}{n'^2}\right). \end{aligned} \tag{2.5}$$

*All other functions in (2.3–2.4) are parametrized by $n(x_3)$*

$$\begin{aligned} g(x_3) &= \frac{d}{n(x_3)^2}, \qquad m(x_3) = -\frac{n'(x_3)}{\alpha}, \qquad \ell(x_3) = \frac{n(x_3)n''(x_3)}{n'(x_3)}, \\ f(x_3) &= 1 - 3\alpha^2 - \alpha\frac{3x_3 m(x_3) - 2\big(a^2 - x_3^2\big)m'(x_3)}{n(x_3)} + \frac{x_3\ell(x_3) - \big(a^2 - x_3^2\big)\ell'(x_3)}{n(x_3)}. \end{aligned} \tag{2.6}$$

*Here $d$ is arbitrary numerical parameter and $z' = \partial z/\partial x_3$.*

The proof is straightforward.

In this note we consider particular solutions of differential equation (2.5) only. Namely, substituting the following ansatz:

$$n(x_3) = c(x_3 + e)^\beta, \qquad c, e, \beta \in \mathbb{R}, \tag{2.7}$$

in (2.5) one gets a system of the algebraic equations on the three parameters $\alpha$, $\beta$ and $e$ whereas two other parameters $c$ and $d$ remain free.

**Theorem 2.** *Differential equation (2.5) has five particular solutions in the form of (2.7) only:*

$$\begin{array}{llll} 1. & \pm\alpha = \beta = 1, & e = 0, & n(x_3) = cx_3, \\ 2. & \pm\alpha = \beta = \frac{1}{3}, & e = 0, & n(x_3) = cx_3^{1/3}, \\ 3. & \pm\alpha = \beta = \frac{1}{6}, & e = a, & n(x_3) = c(x_3 + a)^{1/6}, \\ 4. & \pm\alpha = \beta = \frac{1}{2}, & e \in \mathbb{R}, & n(x_3) = c(x_3 + e)^{1/2}, \\ 5. & \pm\alpha = \beta = \frac{1}{4}, & e = a, & n(x_3) = c(x_3 + a)^{1/4}. \end{array} \tag{2.8}$$

*The corresponding Hamilton functions (2.3) are equal to*

$$\begin{aligned} H_1 &= J_1^2 + J_2^2 + 4J_3^2 + cx_1 + \frac{d}{x_3^2}, \\ H_2 &= J_1^2 + J_2^2 + \frac{4}{3}J_3^2 + \frac{cx_1}{x_3^{2/3}} + \frac{d}{x_3^{2/3}} \\ H_3 &= J_1^2 + J_2^2 + \left(\frac{7}{12} + \frac{x_3}{2(x_3 + a)}\right) J_3^2 + \frac{cx_1}{(x_3 + a)^{5/6}} + \frac{d}{(x_3 + a)^{1/3}}, \\ H_4 &= J_1^2 + J_2^2 + \left(1 + \frac{x_3}{x_3 + e} - \frac{x_3^2 - a^2}{4(x_3 + e)^2}\right) J_3^2 + \frac{cx_1}{(x_3 + e)^{1/2}} + \frac{d}{x_3 + e}, \\ H_5 &= J_1^2 + J_2^2 + \left(\frac{13}{16} + \frac{3x_3}{8(x_3 + a)}\right) J_3^2 + \frac{cx_1}{(x_3 + a)^{3/4}} + \frac{d}{(x_3 + a)^{1/2}}. \end{aligned} \tag{2.9}$$

*Transformation $\alpha \to -\alpha$ leads to the transformation of the free parameters $(c, d) \to (-c, -d)$.*

Explicit expressions for additional cubic integrals of motion $K_1, \ldots, K_5$ may be obtained by using definition (2.4) and equations (2.6) and (2.8).

The Hamilton function $H_1$ describes the Goryachev–Chaplygin top [2]. The second integrable system with Hamiltonian $H_2$ was found by Goryachev [3]. The Hamilton function $H_4$ and the corresponding cubic integral of motion $K_4$ was studied by Dullin and Matveev [4]. The third and fifth integrable systems with Hamiltonians $H_3$ and $H_5$ are new.

At present we do not know whether our systems in implicit or explicit forms (2.5)–(2.9) overlap with the families of integrable geodesic flows on $S^2$ considered by Selivanova [5] and Kiyohara [6]. Recall that in [5, 6] all the geodesic flows are defined in implicit form only (see also the discussion in [4]).

## 3. The Lax matrices

In the fourth case (2.8) and (2.9) the parameter $e$ is an additional free parameter which introduces some differences between the systems. In order to study common properties of all the five systems listed in theorem 2 we formally put $e = 0$ below.

Let us introduce the $2 \times 2$ Hermitian matrix

$$T(\lambda) = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}(\lambda),$$

where $\lambda$ is a spectral parameter and

$$\begin{aligned} A(\lambda) &= (\lambda - q_1)(\lambda - q_2) = \lambda^2 - 2\lambda\alpha J_3 + (\alpha^2 - f(x_3))J_3^2 - J_1^2 - J_2^2 - g(x_3), \\ B(\lambda) &= (x_1 + \mathrm{i}x_2)m(x_3)\lambda + J_3(x_1 + \mathrm{i}x_2)\ell(x_3) + (J_1 + \mathrm{i}J_2)n(x_3), \\ D(\lambda) &= -n(x_3)^2. \end{aligned} \tag{3.1}$$

The trace of this matrix

$$t(\lambda) = A(\lambda) + D(\lambda) = \lambda^2 - \lambda H_L + K_L$$

gives rise to integrals of motion in the involution for the generalized Lagrange system

$$H_L = 2\alpha J_3, \qquad K_L = (\alpha^2 - f(x_3))J_3^2 - J_1^2 - J_2^2 - g(x_3) - n(x_3)^2.$$

The corresponding equations of motion may be rewritten in the form of the Lax triad

$$\frac{\mathrm{d}}{\mathrm{d}t}T(\lambda) = [T(\lambda), M(\lambda)] + N(\lambda), \qquad \mathrm{tr}\, N(\lambda) = 0.$$

In contrast with the Lax pair equations at $N(\lambda) = 0$, in the generic case the determinant $\Delta(\lambda) = \det T(\lambda)$ of the matrix $T(\lambda)$ (3.1) is a dynamical function which does not commute with integrals of motion:

$$\begin{aligned} \beta &= 1 & \Delta(\lambda) &= -\frac{a^2}{\beta^2}\lambda^2\left(\frac{\partial n(x_3)}{\partial x_3}\right)^2 + d, \\ \beta &= \frac{1}{3} & \Delta(\lambda) &= -\frac{a^2}{\beta^2}(\lambda + q_1 + q_2)^2\left(\frac{\partial n(x_3)}{\partial x_3}\right)^2 + d, \\ \beta &= \frac{1}{6} & \Delta(\lambda) &= -\frac{a}{\beta}(\lambda + q_1 + q_2)^2\left(\frac{\partial n^2(x_3)}{\partial x_3}\right) + d, \\ \beta &= \frac{1}{2} & \Delta(\lambda) &= -\frac{a^2}{\beta^2}\lambda(\lambda + q_1 + q_2)\left(\frac{\partial n(x_3)}{\partial x_3}\right)^2 + d, \\ \beta &= \frac{1}{4} & \Delta(\lambda) &= -\frac{a}{\beta}\lambda(\lambda + q_1 + q_2)\left(\frac{\partial n^2(x_3)}{\partial x_3}\right) + d. \end{aligned} \tag{3.2}$$

At $\pm\alpha = \beta = 1$ and $n(x_3) = cx_3$ matrix $T(\lambda)$ (3.1) was constructed in [7]. In this case matrix $T(\lambda)$ defines representation of the Sklyanin algebra on the space $T^*S^2$ associated with the symmetric Neumann system [7].

**Theorem 3.** *If $n(x_3)$ is one of the particular solutions (2.8) of the differential equations (2.5) then $T(\lambda)$ (3.1) satisfies the following deformation of the Sklyanin algebra*

$$\{\overset{1}{T}(\lambda), \overset{2}{T}(\mu)\} = [r(\lambda - \mu), \overset{1}{T}(\lambda)\, \overset{2}{T}(\mu)] + Z(\lambda, \mu), \tag{3.3}$$

*where* $\overset{1}{T}(\lambda) = T(\lambda) \otimes I$, $\overset{2}{T}(\mu) = I \otimes T(\mu)$, $I$ *is a unit matrix and*

$$r(\lambda-\mu) = \frac{2\mathrm{i}\alpha}{\lambda-\mu}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}. \tag{3.4}$$

*Deformation* $Z(\lambda,\mu)$ *is the Hermitian matrix*

$$Z(\lambda,\mu) = \begin{pmatrix}0 & u(\mu) & -u(\lambda) & 0\\ u^*(\mu) & 0 & w(\lambda,\mu) & 0\\ -u^*(\lambda) & w^*(\lambda,\mu) & 0 & 0\\ 0&0&0&0\end{pmatrix}, \tag{3.5}$$

*which depends on the entries of* $T(\lambda)$ *(3.1) only:*

- *at* $\beta = 1$ *we have* $u = w = 0$*;*
- *at* $\beta = \frac{1}{3}, \frac{1}{6}$ *we have*

$$u(\mu) = -4\mathrm{i}\alpha\frac{\sqrt{\Delta(\mu)-d}}{D(\mu)}\frac{\sqrt{\Delta(\mu)-d}\,B(\lambda) - \sqrt{\Delta(\lambda)-d}\,B(\mu)}{\lambda-\mu},$$

$$w(\lambda,\mu) = -4\mathrm{i}\alpha\frac{\Delta(\lambda)-\Delta(\mu)}{\lambda-\mu};$$

- *at* $\beta = \frac{1}{2}, \frac{1}{4}$ *we have*

$$u(\mu) = -2\mathrm{i}\alpha\frac{\Delta(\mu)-d}{\mu D(\mu)}\frac{\mu B(\lambda)-\lambda B(\mu)}{\lambda-\mu},$$

$$w(\lambda,\mu) = -2\mathrm{i}\alpha\frac{\Delta(\lambda)-\Delta(\mu)}{\lambda-\mu}.$$

*Here* $\Delta(\lambda) = A(\lambda)\Delta(\lambda) - B(\lambda)B^*(\lambda)$ *is the determinant of the matrix* $T(\lambda)$ *(3.1).*

The proof is straightforward.

One of the main properties of the Sklyanin algebra is that for *any* numerical matrices $\mathcal{K}$ and for some *special* dynamical matrices $\mathcal{K}$ coefficients of the trace of the matrix $\mathscr{L}(\lambda) = \mathcal{K}T(\lambda)$ give rise the commutative subalgebra

$$\{\mathrm{tr}\,\mathcal{K}T(\lambda), \mathrm{tr}\,\mathcal{K}T(\mu)\} = 0,$$

(see [8] and references therein). All the generators of this subalgebra are linear polynomials on coefficients of entries $T_{ij}(\lambda)$, which are interpreted as integrals of motion for integrable system associated with matrices $T(\lambda)$ and $\mathcal{K}$ [8].

Deformation of the Sklyanin algebra (3.3) and (3.5) has the same property.

**Theorem 4.** *If dynamical matrix* $\mathcal{K}$ *has the form*

$$\mathcal{K} = \begin{pmatrix}\lambda + 2\alpha J_3 & b_1\\ c_1 & 0\end{pmatrix} \qquad b_1, c_1 \in \mathbb{C},$$

*then coefficients of the polynomial*

$$P(\lambda) \equiv \mathrm{tr}\,\mathcal{K}T(\lambda) = \lambda^3 - \lambda H + K \tag{3.6}$$

*are in the involution on* $T^*S^2$.

If $b_1 = c_1 = 1/2$ then the first coefficient $H$ in $P_3(\lambda)$ (3.6) coincides with one of the Hamiltonians $H_1, \dots, H_5$ (2.9) listed in theorem 2, whereas the second coefficients $K$ is the corresponding cubic integral $K_1, \dots, K_5$ (2.4). If $b_1$ and $c_1$ is arbitrary one gets the same Hamiltonians up to the suitable rescaling of $x$ and rotations

$$x \to bUx, \qquad J \to UJ, \tag{3.7}$$

where $b$ is numerical parameter and $U$ is orthogonal constant matrix.

The equations of motion associated with the Hamilton function $H$ (3.6) may be rewritten as a Lax triad for the matrix $\mathscr{L}(\lambda) = \mathcal{K}T(\lambda)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}(\lambda) = [\mathscr{L}(\lambda), \mathscr{M}(\lambda)] + \mathscr{N}(\lambda), \qquad \operatorname{tr}\mathscr{N}(\lambda) = 0.$$

Here matrices $\mathscr{M}$ and $\mathscr{N}$ are restored from the deformed algebra (3.3) and definition of Hamiltonian (3.6) in just the same way as for the usual Sklyanin algebra [7].

At $e \neq 0$ in the fourth case (2.8) and (2.9) we have more complicated deformation of the Sklyanin algebra, which will be studied separately.

## 4. Isomorphism of the systems at $a = 0$

For all the considered systems (2.9) at $a = 0$ the additional term $Z(\lambda, \mu)$ in (3.3) is equal to zero according to (3.2) and (3.5)

$$a = 0 \quad \Longrightarrow \quad \Delta(\lambda) = d \quad \Longrightarrow \quad Z(\lambda, \mu) = 0.$$

In this case matrices $T(\lambda)$ associated with five integrable systems (2.9) define five representations of the Sklyanin algebra on the space $T^*S^2$. Of course, these representations are related to each other by canonical transformations.

**Theorem 5.** *At $a = 0$, i.e. on the non-physical sphere $S^2$ with zero radius, integrable systems listed in the theorem 2 are isomorphic to each other.*

To prove this theorem we introduce the variables

$$\begin{aligned} p_j &= \frac{1}{2\alpha \mathrm{i}} \ln B(q_j) \\ &= \frac{1}{2\alpha \mathrm{i}} \ln(q_j(x_1 + \mathrm{i}x_2)m(x_3) + J_3(x_1 + \mathrm{i}x_2)\ell(x_3) + (J_1 + \mathrm{i}J_2)n(x_3)). \end{aligned} \tag{4.1}$$

At $a = 0$ variables $p_{1,2}$ and $q_{1,2}$ are canonical Darboux variables according to (3.3)

$$\{p_i, q_j\} = \delta_{ij}, \qquad \{p_i, p_j\} = \{q_i, q_j\} = 0, \qquad i, j = 1, 2.$$

In order to construct canonical transformations which relate integrable systems with Hamiltonians $H_1, \dots, H_5$ (2.9) we have to identify variables $p_{1,2}$, $q_{1,2}$ (2.1) and (4.1) associated with the different functions $n(x_3)$ (2.8).

We could not lift these symplectic transformations to the Poisson maps. So, we cannot assert that integrable systems (2.9) are isomorphic on the generic symplectic leaves (1.2).

The result of theorem 5 may be interpreted in the following way. At $a = 0$ on the special symplectic leaf of the Lie algebra $e(3)$ there exists a germ of single integrable system with Hamiltonian $H$ (1.3). Using canonical symplectic transformations one can get infinitely many different forms of this integrable system. However, according to the theorem 2, these different forms of the germ admit only a denumerable set of the continuation on the generic symplectic leaves with the conservation of the integrability property. A similar observation for another family of integrable systems on the sphere is discussed in [9].

## 5. Summary

Using the separation of variables for the Goryachev–Chaplygin top we constructed a family of integrable systems on the sphere with a cubic additional integral of motion. On the non-physical sphere $S^2$ with zero radius these systems are isomorphic to each other.

On this non-physical sphere at $a = 0$ the separated variables for all five systems coincide with the separated variables for the Goryachev–Chaplygin top up to symplectic transformations. It allows us to integrate equations of motion in quadratures. On the usual sphere at $a \neq 0$ the separated variables are unknown. We suppose that these variables may be constructed using the proposed deformation of the Sklyanin algebra.

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## References

[1] Goryachev D N 1900 *Mat. Sb. Krugka Ljubiteley Mat. Nauk.* **21** 431
[2] Chaplygin S A 1904 *Tr. Otd. Fizicheskih Nauk. Imperatorskogo Obshchestva Lyubitelej Estestvoznaniya* **12** 1
[3] Goryachev D N 1915 *Warsaw Univ. Izv.* **3** 1
[4] Dullin H R and Matveev V S 2004 *Preprint* arxiv:math.DS/0406209 to appear in *Math. Res. Lett.*
[5] Selivanova E N 1999 *Commun. Math. Phys.* **207** 641
[6] Kiyohara K 2001 *Math. Ann.* **320** 487
[7] Kuznetsov V B and Tsiganov A V 1989 *J. Phys.* **22** L73
[8] Sokolov V V and Tsiganov A V 2002 *Theor. Math. Phys.* **131** 118
[9] Kostko A L and Tsiganov A V 2003 *Reg. Chaot. Dyn.* **8** 143